Intersections of Tautological Classes on Blowups of Moduli Spaces of Genus-One Curves

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Abstract

We give two recursions for computing top intersections of tautological classes on blowups of moduli spaces of genus-one curves. One of these recursions is analogous to the well-known string equation. As shown in previous papers, these numbers are useful for computing genus-one enumerative invariants of projective spaces and Gromov-Witten invariants of complete intersections.

Contents

1 Introduction

Moduli spaces of stable curves and stable maps play a prominent role in algebraic geometry, symplectic topology, and string theory. Many geometric results have been obtained by utilizing the fact that the moduli space $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ of degree-d stable maps from genus-zero curves with k marked points into $\mathbb{P}^n$ is a smooth unidimensional orbi-variety of the expected dimension. This is not the case for positive-genus moduli spaces $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$. However, if $d\geq 1$, the closure

$$\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\subset\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$$

of the space $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$ of stable maps with smooth domains is an irreducible orbi-variety of the expected dimension. This component of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$ contains all the relevant genus-one information

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for the purposes of enumerative geometry and, as shown in [LZ] and [Z], of the Gromov-Witten theory.

For $d \ge 3$, $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ is singular. A desingularization of the space $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$, i.e. a smooth orbi-variety $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ and a map

$$\pi\colon \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d),$$

which is biholomorphic onto $\mathfrak{M}^0_{1,k}(\mathbb{P}^n,d)$, is constructed in [VZ]. Via this desingularization and the classical localization theorem of [AB], intersections of naturally arising cohomology classes on $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ can be expressed in terms of integrals of certain ψ-classes on moduli spaces of genus-zero and genus-one stable curves and on blowups of moduli spaces of genus-one stable curves; see below for more details. The former can be computed through two well-known recursions, called string and dilaton equations; see Section 26.3 in [H]. In this paper we obtain two recursions which can be used to express the latter numbers in terms of the former ones; see Theorem 1.1 below. One of these recursions generalizes the genus-one string relation.

If J is a finite nonempty set, let $\overline{\mathcal{M}}_{1,J}$ be the moduli space of genus-one curves with marked points indexed by the set J. Let

$$\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{1,J}$$

be the Hodge line bundle of holomorphic differentials. For each $j \in J$, we denote by

$$L_j \longrightarrow \overline{\mathcal{M}}_{1,J}$$

the universal tangent line for the jth marked point and put

$$\psi_j = c_1(L_j^*) \in H^*\big(\overline{\mathcal{M}}_{1,J};\mathbb{Q}\big).$$

If $(c_j)_{j\in J}$ is a tuple of integers, let

$$\big\langle (c_j)_{j\in J}\big\rangle_{|J|} = \Big\langle \prod_{j\in J}\psi_j^{c_j}, \overline{\mathcal{M}}_{1,J}\Big\rangle.$$

Let I and J be two finite sets, not both empty. The inductive procedure of Subsection **??** in [VZ], which is reviewed in Subsection 2.1 below, constructs a blowup

$$\pi\colon \widetilde{\mathcal{M}}_{1,(I,J)} \longrightarrow \overline{\mathcal{M}}_{1,I\sqcup J}$$

of $\overline{\mathcal{M}}_{1,I\sqcup J}$ along natural subvarieties and their proper transforms. In addition, it describes $|I|+1$ line bundles

$$\tilde{\mathbb{E}},\, \tilde{L}_i \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J)}, \qquad i\in I,$$

and $|I|$ nowhere vanishing sections

$$\tilde{s}_i \in \Gamma\big(\widetilde{\mathcal{M}}_{1,(I,J)}; \tilde{L}_i^*\otimes\tilde{\mathbb{E}}^*\big), \qquad i\in I.$$

These line bundles are obtained by twisting $\mathbb{E}$ and L_i. Since the sections $\tilde{s}_i$ do not vanish, all $|I|+1$ line bundles $\tilde{L}_i$ and $\tilde{\mathbb{E}}^*$ are explicitly isomorphic. They will be denoted by

$$\mathbb{L} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J)}$$

and called the universal tangent bundle. Let

$$\tilde{\psi} = c_1(\mathbb{L}^*) \in H^2\big(\widetilde{\mathcal{M}}_{1,(I,J)};\mathbb{Q}\big)$$

be the corresponding "ψ-class" on $\widetilde{\mathcal{M}}_{1,(I,J)}$. If $(\tilde{c},(c_j)_{j\in J})$ is a tuple of integers, we put

$$\big\langle \tilde{c};(c_j)_{j\in J}\big\rangle_{(|I|,|J|)} = \Big\langle \tilde{\psi}^{\tilde{c}} \cdot \prod_{j\in J} \pi^*\psi_j^{c_j}, \widetilde{\mathcal{M}}_{1,(I,J)}\Big\rangle. \tag{1.1}$$

If $\sum_{j\in J} c_j \neq |J|$, $\tilde{c}<0$, or $c_j<0$ for some $j\in J$, we define this number to be zero.

Theorem 1.1 *Suppose I and J are finite sets, such that $|I|+|J|\geq 2$, and $(\tilde{c},(c_j)_{j\in J})$ is a tuple of integers. If $c_{j^*}=0$ for some $j^*\in J$,*

$$\big\langle \tilde{c};(c_j)_{j\in J}\big\rangle_{(|I|,|J|)} = |I|\big\langle \tilde{c}-1;(c_j)_{j\in J-\{j^*\}}\big\rangle_{(|I|,|J|-1)} + \sum_{j\in J-\{j^*\}} \big\langle \tilde{c};c_j-1,(c_{j'})_{j'\in J-\{j^*,j\}}\big\rangle_{(|I|,|J|-1)}.$$

If $I\neq\emptyset$ and $c_j>0$ for all $j\in J$,

$$\big\langle \tilde{c};(c_j)_{j\in J}\big\rangle_{(|I|,|J|)} = \big\langle \tilde{c};(c_j)_{j\in J}\big\rangle_{(|I|-1,|J|+1)}.$$

Corollary 1.2 *If I and J are finite sets and $I\neq\emptyset$, then*

$$\big\langle \tilde{\psi}^{|I|+|J|}, \widetilde{\mathcal{M}}_{I,J}\big\rangle = \frac{1}{24}\cdot |I|^{|J|}\cdot(|I|-1)!$$

We recall that

$$\big\langle \psi, \overline{\mathcal{M}}_{1,1}\big\rangle = \frac{1}{24}.$$

Thus, Corollary 1.2 follows from Theorem 1.1 by applying the first recursion $|J|$ times and then the second recursion followed by the first $|I|-1$ times.

It is immediate from the construction of Subsection 2.1 below that

$$I=\emptyset \qquad \Longrightarrow \qquad \widetilde{\mathcal{M}}_{1,(I,J)} = \overline{\mathcal{M}}_{1,I\sqcup J} \quad \text{and} \quad \tilde{\psi} = \lambda \equiv c_1(\mathbb{E}).$$

Thus, the two recursions of Theorem 1.1, along with the string and dilaton equations, provide a straightforward algorithm for computing all numbers (1.1).

$$
\begin{tikzcd}
\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \arrow[r, dashed, "\tilde{\iota}"] \arrow[d, "\pi"'] & \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^m,d) \arrow[d, "\pi"'] \\
\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d) \arrow[r, "\iota"] & \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^m,d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \arrow[r, dashed, "\tilde{f}"] \arrow[d, "\pi"'] & \widetilde{\mathfrak{M}}^0_{1,k-1}(\mathbb{P}^n,d) \arrow[d, "\pi"'] \\
\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d) \arrow[r, "f"] & \overline{\mathfrak{M}}_{1,k-1}(\mathbb{P}^n,d)
\end{tikzcd}
$$

Figure 1: Some Natural Properties of $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$

Note that if we take $\tilde{c}=0$ in the first equation of Theorem 1.1, we recover the genus-one string recursion for ψ-classes. This equation is proved in Subsection 2.2 by an argument similar to the standard proof of the string recursion. In particular, we consider the forgetful morphism

$$f\colon \overline{\mathcal{M}}_{1,I\sqcup J} \longrightarrow \overline{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}.$$

We show in Subsection 3.3 that it lifts to a morphism on the blowups,

$$\tilde{f}\colon \widetilde{\mathcal{M}}_{1,(I,J)} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})}.$$

The first recursion of Theorem 1.1 is obtained by comparing $\tilde{\psi}$ with $\tilde{f}^*\tilde{\psi}$. On the other hand, the second equation of Theorem 1.1 follows easily from the relevant definitions, which are reviewed in Subsection 2.1. The reason is that the blowups of $\overline{\mathcal{M}}_{1,I\sqcup J}$ corresponding to the two sides of this equation differ by blowups along loci on which $\prod_{j\in J}\psi_j$ vanishes; see the end of Subsection 2.1.

In [VZ], it is observed that if $n<m$, the natural embedding

$$\iota\colon \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^m,d)$$

induced by the inclusion $\mathbb{P}^n \longrightarrow \mathbb{P}^m$ lifts to an embedding on the desingularizations:

$$\tilde{\iota}\colon \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^m,d).$$

Proceeding analogously to Subsection 3.3, one can show that if $k>0$ the forgetful morphism

$$f\colon \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}_{1,k-1}(\mathbb{P}^n,d)$$

lifts to a morphism

$$\tilde{f}\colon \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \widetilde{\mathfrak{M}}^0_{1,k-1}(\mathbb{P}^n,d).$$

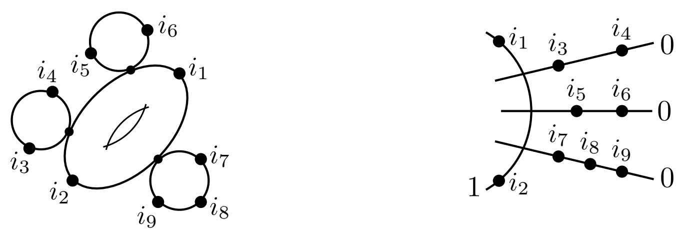
Thus, the desingularization $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ constructed in [VZ] respects at least two properties that play a central role in the Gromov-Witten theory; see Figure 1.

2 Preliminaries

2.1 Blowup Construction

If I is a finite set, let

$$\mathcal{A}_1(I) = \Big\{ \big(I_P, \{I_k\colon k\in K\}\big)\colon K\neq\emptyset;\ I = \bigsqcup_{k\in\{P\}\sqcup K} I_k;\ |I_k|\geq 2\ \forall\, k\in K \Big\}. \tag{2.1}$$



$I_P = \{i_1, i_2\}$
$K = \{1, 2, 3\}$
$I_1 = \{i_3, i_4\}$
$I_2 = \{i_5, i_6\}$
$I_3 = \{i_7, i_8, i_9\}$

Figure 2: A Typical Element of $\overline{\mathcal{M}}_{1,\rho}$

Here P stands for "principal" (component). If $\rho = (I_P, \{I_k : k \in K\})$ is an element of $\mathcal{A}_1(I)$, we denote by $\mathcal{M}_{1,\rho}$ the subset of $\overline{\mathcal{M}}_{1,I}$ consisting of the stable curves $\mathcal{C}$ such that

(i) $\mathcal{C}$ is a union of a smooth torus and $|K|$ projective lines, indexed by K;

(ii) each line is attached directly to the torus;

(iii) for each $k \in K$, the marked points on the line corresponding to k are indexed by I_k.

Let $\overline{\mathcal{M}}_{1,\rho}$ be the closure of $\mathcal{M}_{1,\rho}$ in $\overline{\mathcal{M}}_{1,I}$. Figure 2 illustrates this definition, from the points of view of symplectic topology and of algebraic geometry. In the first diagram, each circle represents a sphere, or $\mathbb{P}^1$. In the second diagram, the irreducible components of $\mathcal{C}$ are represented by curves, and the integer next to each component shows its genus. It is well-known that each space $\overline{\mathcal{M}}_{1,\rho}$ is a smooth subvariety of $\overline{\mathcal{M}}_{1,I}$.

We define a partial ordering on the set $\mathcal{A}_1(I) \sqcup \{(I, \emptyset)\}$ by setting

$$\rho' \equiv \big(I'_P, \{I'_k : k \in K'\}\big) \prec \rho \equiv \big(I_P, \{I_k : k \in K\}\big) \tag{2.2}$$

if $\rho' \neq \rho$ and there exists a map $\varphi\colon K \longrightarrow K'$ such that $I_k \subset I'_{\varphi(k)}$ for all $k \in K$. This condition means that the elements of $\mathcal{M}_{1,\rho'}$ can be obtained from the elements of $\mathcal{M}_{1,\rho}$ by moving more points onto the bubble components or combining the bubble components; see Figure 3.

Let I and J be finite sets such that I is not empty and $|I| + |J| \geq 2$. We put

$$\mathcal{A}_1(I, J) = \Big\{\big((I_P \sqcup J_P), \{I_k \sqcup J_k : k \in K\}\big) \in \mathcal{A}_1(I \sqcup J) : \; I_k \neq \emptyset \; \forall\, k \in K\Big\}.$$

We note that if $\varrho \in \mathcal{A}_1(I \sqcup J)$, then $\varrho \in \mathcal{A}_1(I, J)$ if and only if every bubble component of an element of $\mathcal{M}_{1,\varrho}$ carries at least one element of I. The partially ordered set $(\mathcal{A}_1(I, J), \prec)$ has a unique minimal element

$$\varrho_{\min} \equiv \big(\emptyset, \{I \sqcup J\}\big).$$

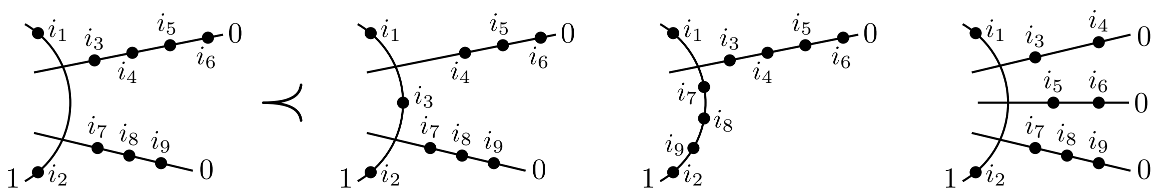


Figure 3: Examples of Partial Ordering (2.2)

Let $<$ be an ordering on $\mathcal{A}_1(I,J)$ extending the partial ordering $\prec$. We denote the corresponding maximal element by $\varrho_{\max}$. If $\varrho\in\mathcal{A}_1(I,J)$, we put

$$\varrho-1=\begin{cases}\max\{\varrho'\in\mathcal{A}_1(I,J)\colon \varrho'<\varrho\}, & \text{if } \varrho\neq\varrho_{\min};\\ 0, & \text{if } \varrho=\varrho_{\min},\end{cases} \tag{2.3}$$

where the maximum is taken with respect to the ordering $<$.

The starting data for the blowup construction of Subsection **??** in [VZ] is given by

$$\overline{\mathcal{M}}_{1,(I,J)}^0=\overline{\mathcal{M}}_{1,I\sqcup J},\qquad \overline{\mathcal{M}}_{1,\varrho}^0=\overline{\mathcal{M}}_{1,\varrho}\quad\forall\,\varrho\in\mathcal{A}_1(I,J),$$
$$\mathbb{E}_0=\mathbb{E}\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}^0,\qquad\text{and}\qquad L_{0,i}=L_i\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}^0\quad\forall\,i\in I.$$

Suppose $\varrho\in\mathcal{A}_1(I,J)$ and we have constructed

($I1$) a blowup $\pi_{\varrho-1}\colon\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}^0$ of $\overline{\mathcal{M}}_{1,(I,J)}^0$ such that $\pi_{\varrho-1}$ is one-to-one outside of the preimages of the spaces $\overline{\mathcal{M}}_{1,\varrho'}^0$ with $\varrho'\le\varrho-1$;

($I2$) line bundles $L_{\varrho-1,i}\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$ for $i\in I$ and $\mathbb{E}_{\varrho-1}\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$.

For each $\varrho^*>\varrho-1$, let $\overline{\mathcal{M}}_{1,\varrho^*}^{\varrho-1}$ be the proper transform of $\overline{\mathcal{M}}_{1,\varrho^*}^0$ in $\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$.

If $\varrho\in\mathcal{A}_1(I,J)$ is as above, let

$$\tilde{\pi}_\varrho\colon\overline{\mathcal{M}}_{1,(I,J)}^{\varrho}\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$$

be the blowup of $\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$ along $\overline{\mathcal{M}}_{1,\varrho}^{\varrho-1}$. We denote by $\overline{\mathcal{M}}_{1,\varrho}^{\varrho}$ the corresponding exceptional divisor. If $\varrho^*>\varrho$, let $\overline{\mathcal{M}}_{1,\varrho^*}^{\varrho}\subset\overline{\mathcal{M}}_{1,(I,J)}^{\varrho}$ be the proper transform of $\overline{\mathcal{M}}_{1,\varrho^*}^{\varrho-1}$. If

$$\varrho=\big((I_P\sqcup J_P),\{I_k\sqcup J_k\colon k\in K\}\big)\in\mathcal{A}_1(I\sqcup J)\qquad\text{and}\qquad i\in I,$$

we put

$$L_{\varrho,i}=\begin{cases}\tilde{\pi}_\varrho^*L_{\varrho-1,i}, & \text{if } i\notin I_P;\\ \tilde{\pi}_\varrho^*L_{\varrho-1,i}\otimes\mathcal{O}(-\overline{\mathcal{M}}_{1,\varrho}^{\varrho}), & \text{if } i\in I_P;\end{cases}\qquad \mathbb{E}_\varrho=\tilde{\pi}_\varrho^*\,\mathbb{E}_{\varrho-1}\otimes\mathcal{O}(\overline{\mathcal{M}}_{1,\varrho}^{\varrho}). \tag{2.4}$$

It is immediate that the requirements ($I1$) and ($I2$), with $\varrho-1$ replaced by ϱ, are satisfied.

We conclude the blowup construction after $|\varrho_{\max}|$ steps. Let

$$\widetilde{\mathcal{M}}_{1,(I,J)}=\overline{\mathcal{M}}_{1,(I,J)}^{\varrho_{\max}},\qquad \tilde{L}_i=L_{\varrho_{\max},i}\ \ \forall\,i\in I,\qquad \tilde{\mathbb{E}}=\mathbb{E}_{\varrho_{\max}}.$$

By Lemma **??** in [VZ], the end result of this blowup construction is well-defined, i.e. independent of the choice of an ordering $<$ extending the partial ordering $\prec$. The reason is that different extensions of the partial order $\prec$ correspond to different orders of blowups along disjoint subvarieties. By the inductive assumption ($I4$) in Subsection **??** of [VZ], there is a natural isomorphism between the line bundles $\tilde{L}_i$ and $\tilde{\mathbb{E}}^*$. Thus, these line bundles are the same. We denote them by $\mathbb{L}$.

We are now ready to verify the second equation in Theorem 1.1. If $i^*\in I$,

$$\mathcal{A}_1\big(I-\{i^*\},J\sqcup\{i^*\}\big)\subset\mathcal{A}_1(I,J)\qquad\text{and}$$
$$\mathcal{A}_1(I,J)-\mathcal{A}_1\big(I-\{i^*\},J\sqcup\{i^*\}\big)=\big\{\varrho=\big(I_P\sqcup J_P,\big\{\{i^*\}\sqcup J_1\big\}\sqcup\{I_k\sqcup J_k\colon k\in K'\}\big)\in\mathcal{A}_1(I\sqcup J)\big\}.$$

With ϱ as above, we have a natural isomorphism

$$\overline{\mathcal{M}}_{1,\varrho} \approx \overline{\mathcal{M}}_{1,\bar{\varrho}} \times \overline{\mathcal{M}}_{0,\{q,i^*\}\sqcup J_1}, \qquad \text{where} \quad \bar{\varrho} = \big(I_P \sqcup J_P \sqcup \{p\}, \{I_k \sqcup J_k : k \in K'\}\big).$$

Let

$$\pi_2 : \overline{\mathcal{M}}_{1,\varrho} \longrightarrow \overline{\mathcal{M}}_{0,\{q,i^*\}\sqcup J_1}$$

be the projection map. By definition,

$$\psi_j\big|_{\overline{\mathcal{M}}_{1,\varrho}} = \pi_2^*\psi_j \qquad \forall\, j \in J_1 \qquad \Longrightarrow \qquad \prod_{j\in J_1}\psi_j\big|_{\overline{\mathcal{M}}_{1,\varrho}} = \pi_2^*\prod_{j\in J_1}\psi_j = \pi_2^* 0 = 0,$$

since the dimension of $\overline{\mathcal{M}}_{0,\{q,i^*\}\sqcup J_1}$ is $|J_1|-1$. It follows that

$$\prod_{j\in J}\psi_j\big|_{\overline{\mathcal{M}}_{1,\varrho}} = 0 \qquad \forall\, \varrho \in \mathcal{A}_1(I,J) - \mathcal{A}_1\big(I-\{i^*\}, J\sqcup\{i^*\}\big).$$

Thus, the constructions of $\tilde{\psi} \equiv c_1(\tilde{\mathbb{E}})$ from $\lambda \equiv c_1(\mathbb{E}_0)$ for $\widetilde{\mathcal{M}}_{1,(I-\{i^*\},J\sqcup\{i^*\})}$ and $\widetilde{\mathcal{M}}_{1,(I,J)}$ differ by varieties along which $\prod_{j\in J}\psi_j^{c_j}$ vanishes, as long as $c_j > 0$ for all $j \in J$. We conclude that

$$\Big\langle \tilde{\psi}^{\tilde{c}} \cdot \prod_{j\in J}\pi^*\psi_j^{c_j}, \widetilde{\mathcal{M}}_{1,(I,J)} \Big\rangle = \Big\langle \tilde{\psi}^{\tilde{c}} \cdot \prod_{j\in J}\pi^*\psi_j^{c_j}, \widetilde{\mathcal{M}}_{1,(I-\{i^*\},J\sqcup\{i^*\})} \Big\rangle$$

whenever $c_j > 0$ for all $j \in J$, as needed.

2.2 Outline of Proof of First Recursion in Theorem 1.1

In this subsection we state three results, Proposition 2.1 and Lemmas 2.2 and 2.3, that lead in a straightforward way to the first recursion of Theorem 1.1. They are proved in the next section.

If I is a finite set and i, j are distinct elements of I, let

$$\rho_{ij} = \big(I - \{i,j\}, \{\{i,j\}\}\big) \in \mathcal{A}_1(I).$$

There is a natural decomposition

$$\overline{\mathcal{M}}_{1,\varrho_{ij}} = \overline{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}} \times \overline{\mathcal{M}}_{0,\{q,i,j\}}. \tag{2.5}$$

The second component is a one-point space. Let

$$\pi_P, \pi_B : \overline{\mathcal{M}}_{1,\varrho_{ij}} \longrightarrow \overline{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}}, \overline{\mathcal{M}}_{0,\{q,i,j\}} \tag{2.6}$$

be the two projection maps. Here P and B stand for "principal" and "bubble" (components). It is immediate that

$$\lambda|_{\overline{\mathcal{M}}_{1,\varrho_{ij}}} = \pi_P^*\lambda \qquad \text{and} \tag{2.7}$$

$$\psi_{j'}\big|_{\overline{\mathcal{M}}_{1,\varrho_{ij}}} = \begin{cases} \pi_P^*\psi_{j'}, & \text{if } j' \neq i, j; \\ \pi_B^*\psi_{j'} = 0, & \text{if } j' = i, j; \end{cases} \qquad \forall\, j' \in I. \tag{2.8}$$

In the $j'=i,j$ case the restriction of $\psi_{j'}$ vanishes because the second component is zero-dimensional.

If I is a finite set, $|I|\geq 2$, and $j^*\in I$, there is a natural forgetful morphism

$$f\colon \overline{\mathcal{M}}_{1,I}\longrightarrow \overline{\mathcal{M}}_{1,I-\{j^*\}}.$$

It is obtained by dropping the marked point j^* from every element of $\overline{\mathcal{M}}_{1,I}$ and contracting the unstable components of the resulting curve. It is straightforward to check that

$$\lambda = f^*\lambda \qquad \text{and} \tag{2.9}$$

$$\psi_j = f^*\psi_j + \overline{\mathcal{M}}_{1,\varrho_{jj^*}} \quad\Longrightarrow\quad f^*\psi_j|_{\overline{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi_P^*\psi_p \qquad \forall\, j\in I-\{j^*\}; \tag{2.10}$$

see Chapter 25 in [H], for example. From (2.8) and (2.10), we find that

$$\psi_j^{c_j} = \psi_j^{c_j-1}\big(f^*\psi_j + \overline{\mathcal{M}}_{1,\varrho_{jj^*}}\big) = f^*\psi_j^{c_j} + \big(\pi_P^*\psi_p^{c_j-1}\big)\cap\overline{\mathcal{M}}_{1,\varrho_{jj^*}} \quad \forall\, j\in I-\{j^*\},\ c_j>0. \tag{2.11}$$

If I and J are finite sets, $i\in I$, and $j\in J$, then $\overline{\mathcal{M}}_{1,\varrho_{ij}}$ is a divisor in $\overline{\mathcal{M}}_{1,I\sqcup J}$. Thus, in the notation of the previous subsection,

$$\overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}} = \overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}-1}.$$

Since ϱ_{ij} is a maximal element of $(A_1(I,J),\prec)$, the blowup loci at the stages of the construction described in Subsection 2.1 that follow the blowup along $\overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}-1}$ are disjoint from $\overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}}$. Thus, we can view $\overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}}$ as a divisor in $\widetilde{\mathcal{M}}_{1,(I,J)}$. We denote it by $\widetilde{\mathcal{M}}_{1,\varrho_{ij}}$. If $i,j\in J$, $\overline{\mathcal{M}}_{1,\varrho_{ij}}$ is also a divisor in $\overline{\mathcal{M}}_{1,I\sqcup J}$. Thus, its proper transform $\overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho}$ in $\overline{\mathcal{M}}_{1,(I,J)}^{\varrho}$ is a divisor for every $\varrho\in\mathcal{A}_1(I,J)$. Let

$$\widetilde{\mathcal{M}}_{1,\varrho_{ij}} = \overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{\max}} \subset \widetilde{\mathcal{M}}_{1,(I,J)}.$$

Proposition 2.1 *Suppose I and J are finite sets such that $|I|+|J|\geq 2$ and $j^*\in J$. If*

$$\pi\colon \widetilde{\mathcal{M}}_{1,(I,J)}\longrightarrow \overline{\mathcal{M}}_{1,I\sqcup J} \qquad \textit{and} \qquad \pi\colon \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})}\longrightarrow \overline{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}$$

are blowups as in Subsection 2.1, the forgetful map

$$f\colon \overline{\mathcal{M}}_{1,I\sqcup J}\longrightarrow \overline{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}$$

lifts to a morphism

$$\tilde{f}\colon \widetilde{\mathcal{M}}_{1,(I,J)}\longrightarrow \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})};$$

see Figure 4. Furthermore,

$$\tilde{\psi} = \tilde{f}^*\tilde{\psi} + \sum_{i\in I}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}.$$

Lemma 2.2 *With notation as in Proposition 2.1, for all $i\in I$*

$$\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} = \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}\times\overline{\mathcal{M}}_{0,\{q,i,j^*\}} \qquad \textit{and}$$

$$\pi_P\circ\pi = \pi\circ\pi_P\colon \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}\longrightarrow \overline{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\})\sqcup(J-\{j^*\})},$$

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_{1,(I,J)} \arrow[r, dashed, "\tilde{f}"] \arrow[d, "\pi"'] & \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})} \arrow[d, "\pi"'] \\
\overline{\mathcal{M}}_{1,I\sqcup J} \arrow[r, "f"] & \overline{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}
\end{tikzcd}$$

Figure 4: Illustration of Main Statement of Proposition 2.1

where

$$\pi_P\colon \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} \longrightarrow \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}$$

is again the projection onto the first component. Furthermore, if $\tilde{\psi}$ *denotes the universal* ψ*-class and* $\tilde{f}$ *is as in Proposition 2.1, then*

$$\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = 0 \qquad \textit{and} \qquad \big(\tilde{f}^*\tilde{\psi}\big)\big|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \pi_P^*\tilde{\psi}.$$

Lemma 2.3 *With notation as in Proposition 2.1, for all* $j\in J-\{j^*\}$

$$\pi^{-1}\big(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\big) = \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \approx \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}} \qquad \textit{and}$$
$$\pi_P\circ\pi = \pi\circ\pi_P\colon \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \longrightarrow \overline{\mathcal{M}}_{1,I\sqcup((J-\{j,j^*\})\sqcup\{p\})},$$

where

$$\pi_P\colon \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}$$

is again the projection onto the first component. Furthermore, if $\tilde{\psi}$ *denotes the universal* ψ*-class on* $\widetilde{\mathcal{M}}_{1,(I,J)}$ *and on* $\widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}$*, then*

$$\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \big(\tilde{f}^*\tilde{\psi}\big)\big|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi_P^*\tilde{\psi}.$$

We are now ready to verify the first identity in Theorem 1.1. We can assume that $\tilde{c}\neq 0$; otherwise, it reduces to the standard genus-one string equation. Note that if $i_1,i_2\in I$ and $i_1\neq i_2$, then

$$\overline{\mathcal{M}}_{1,\varrho_{i_1j^*}} \cap \overline{\mathcal{M}}_{1,\varrho_{i_2j^*}} = \emptyset \qquad \Longrightarrow \qquad \widetilde{\mathcal{M}}_{1,\varrho_{i_1j^*}} \cap \widetilde{\mathcal{M}}_{1,\varrho_{i_2j^*}} = \emptyset. \tag{2.12}$$

Thus, by Proposition 2.1 and Lemma 2.2, applied repeatedly,

$$\tilde{\psi}^{\tilde{c}} = \tilde{\psi}^{\tilde{c}-1}\big(\tilde{f}^*\psi + \sum_{i\in I}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}\big) = \tilde{f}^*\tilde{\psi}^{\tilde{c}} + \sum_{i\in I}\big(\pi_P^*\tilde{\psi}^{\tilde{c}-1}\big)\cap\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}. \tag{2.13}$$

On the other hand, by (2.11) and Lemma 2.3,

$$\pi^*\psi_j^{c_j} = \tilde{f}^*\pi^*\psi_j^{c_j} + \big(\pi_P^*\pi^*\psi_p^{c_j-1}\big)\cap\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \qquad \forall\, j\in J-\{j^*\}. \tag{2.14}$$

If $c_j=0$, we define the last term in (2.14) to be zero. Similarly to (2.12),

$$\overline{\mathcal{M}}_{1,\varrho_{ij^*}} \cap \overline{\mathcal{M}}_{1,\varrho_{jj^*}} = \emptyset \quad \Longrightarrow \quad \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} \cap \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} = \emptyset \qquad \forall\, j\in J-\{j^*\},\ i\in I\sqcup J-\{j,j^*\}. \tag{2.15}$$

Thus, by (2.13), (2.14), and Lemmas 2.2 and 2.3,

$$\begin{aligned}
\big\langle \tilde{c}; (c_j)_{j\in J-\{j^*\}}\big\rangle_{(|I|,|J|)} &\equiv \Big\langle \tilde{\psi}^{\tilde{c}}\cdot \prod_{j\in J-\{j^*\}}\pi^*\psi_j^{c_j}, \widetilde{\mathcal{M}}_{1,(I,J)}\Big\rangle\\
&= \Big\langle \tilde{f}^*\Big(\tilde{\psi}^{\tilde{c}}\cdot \prod_{j\in J-\{j^*\}}\pi^*\psi_j^{c_j}\Big), \widetilde{\mathcal{M}}_{1,(I,J)}\Big\rangle + \sum_{i\in I}\Big\langle \pi_P^*\Big(\tilde{\psi}^{\tilde{c}-1}\cdot \prod_{j\in J-\{j^*\}}\pi^*\psi_j^{c_j}\Big), \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}\Big\rangle\\
&\qquad\qquad + \sum_{j\in J-\{j^*\}}\Big\langle \pi_P^*\Big(\tilde{\psi}^{\tilde{c}}\cdot\pi^*\psi_p^{c_j-1}\cdot \prod_{j'\in J-\{j^*,j\}}\pi^*\psi_{j'}^{c_{j'}}\Big), \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}\Big\rangle\\
&= 0 + \sum_{i\in I}\Big\langle \tilde{\psi}^{\tilde{c}-1}\cdot \prod_{j\in J-\{j^*\}}\pi^*\psi_j^{c_j}, \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}\Big\rangle\\
&\qquad + \sum_{j\in J-\{j^*\}}\Big\langle \tilde{\psi}^{\tilde{c}}\cdot\pi^*\psi_p^{c_j-1}\cdot \prod_{j'\in J-\{j^*,j\}}\pi^*\psi_{j'}^{c_{j'}}, \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}\Big\rangle\\
&\equiv |I|\big\langle \tilde{c}-1; (c_j)_{j\in J-\{j^*\}}\big\rangle_{(|I|,|J|-1)} + \sum_{j\in J-\{j^*\}}\big\langle \tilde{c}; c_j-1, (c_{j'})_{j'\in J-\{j^*,j\}}\big\rangle_{(|I|,|J|-1)}.
\end{aligned}$$

We have thus derived the first identity in Theorem 1.1 from Proposition 2.1 and Lemmas 2.2 and 2.3.

3 Proofs of Main Structural Results

3.1 Proof of Lemma 2.2

Suppose I is a finite set and i,j are distinct elements of I. It is well-known that the normal bundle $\mathcal{N}_{\overline{\mathcal{M}}_{1,I}}\overline{\mathcal{M}}_{1,\varrho_{ij}}$ of $\overline{\mathcal{M}}_{1,\varrho_{ij}}$ in $\overline{\mathcal{M}}_{1,I}$ is given by

$$\mathcal{N}_{\overline{\mathcal{M}}_{1,I}}\overline{\mathcal{M}}_{1,\varrho_{ij}} = \pi_P^*L_p\otimes\pi_B^*L_q = \pi_P^*L_p, \tag{3.1}$$

where π_B and π_B are as in (2.6) and

$$L_p \longrightarrow \overline{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}} \qquad \text{and} \qquad L_q \longrightarrow \overline{\mathcal{M}}_{0,\{q,i,j\}}$$

are the universal tangent line bundles for the marked points p and q; see [P], for example. The last equality in (3.1) is due to the fact that $\overline{\mathcal{M}}_{0,\{q,i,j\}}$ consists of one point.

Suppose in addition that

$$\varrho\equiv\big(I_P,\{I_k\!: k\!\in\! K\}\big)\in\mathcal{A}_1(I) \qquad \text{and} \qquad \varrho\prec\varrho_{ij}. \tag{3.2}$$

Then, by the definition of the partial ordering $\prec$ in (2.2),

$$\{i,j\}\subset I_k \qquad \text{for some} \qquad k\in K.$$

We define $\mu_{ij}(\varrho)\in\mathcal{A}_1\big((I-\{i,j\})\sqcup\{p\}\big)$ by

$$\mu_{ij}(\varrho) = \begin{cases} \big(I_P\sqcup\{p\},\{I_{k'}\!: k'\!\in\! K-\{k\}\}\big), & \text{if } I_k\!=\!\{i,j\};\\ \big(I_P,\big\{(I_k\!-\!\{i,j\})\sqcup\{p\}\big\}\sqcup\{I_{k'}\!: k'\!\in\! K-\{k\}\}\big), & \text{if } I_k\supsetneq\{i,j\}. \end{cases} \tag{3.3}$$

It is straightforward to see that

$$\overline{\mathcal{M}}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}_{1,\varrho} = \overline{\mathcal{M}}_{1,\mu_{ij}(\varrho)}\times\overline{\mathcal{M}}_{0,\{q,i,j\}} \subset \overline{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}. \tag{3.4}$$

Lemma 3.1 *If I and J are finite sets, $i\in I$, and $j\in J$, then the map*

$$\mu_{ij}\colon \big\{\varrho\in\mathcal{A}_1(I,J)\colon \varrho\prec\varrho_{ij}\big\} \longrightarrow \mathcal{A}_1\big((I-\{i\})\sqcup\{p\}, J-\{j\}\big) \tag{3.5}$$

is an isomorphism of partially ordered sets.

This lemma follows easily from (2.2) and (3.3). It implies that given an order $<$ on

$$\mathcal{A}_1\big((I-\{i\})\sqcup\{p\}, J-\{j\}\big)$$

extending the partial ordering $\prec$, we can choose an order $<$ on $A_1(I,J)$ that extends the partial ordering $\prec$ such that

$$\varrho_1,\varrho_2\prec\varrho_{ij},\quad \mu_{ij}(\varrho_1)<\mu_{ij}(\varrho_2) \qquad\Longrightarrow\qquad \varrho_1<\varrho_2.$$

Below we refer to the constructions of Subsection 2.1 for the sets

$$\mathcal{A}_1\big((I-\{i\})\sqcup\{p\}, J-\{j\}\big) \qquad\text{and}\qquad \mathcal{A}_1(I,J)$$

corresponding to such compatible orders $<$. We extend the map μ_{ij} of (3.5) to $\{0\}\sqcup\mathcal{A}_1(I,J)$ by setting

$$\mu_{ij}(\varrho)=\begin{cases}\mu_{ij}(\max\{\varrho'<\varrho:\varrho'\prec\varrho_{ij}\}), & \text{if } \exists\,\varrho'<\varrho \text{ s.t. } \varrho'\prec\varrho_{ij};\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.2 *Suppose I and J are finite sets, $i\in I$, and $j\in J$. If $\varrho\in\mathcal{A}_1(I,J)$ and $\varrho<\varrho_{ij}$, then with notation as in Subsection 2.1 and in (2.5)*

$$\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\}),J-\{j\})}\times\overline{\mathcal{M}}_{0,\{q,i,j\}},$$
$$\mathbb{E}_{\varrho}\big|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}}=\pi_P^*\mathbb{E}_{\mu_{ij}(\varrho)},\qquad \textit{and}\qquad \mathcal{N}_{\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\pi_P^*L_{\mu_{ij}(\varrho),p},$$

where

$$\pi_P\colon\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}\longrightarrow\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}$$

is the projection map onto the first component.

By (2.5), (2.7), and (3.1), Lemma 3.2 holds for $\varrho=0$. Suppose $\varrho\in\mathcal{A}_1(I,J)$, $\varrho<\varrho_{ij}$, and the three claims hold for $\varrho-1$. If $\varrho\not\prec\varrho_{ij}$, then

$$\mu_{ij}(\varrho)=\mu_{ij}(\varrho-1)\qquad\Longrightarrow$$
$$\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}=\overline{\mathcal{M}}^{\mu_{ij}(\varrho-1)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})},\quad \mathbb{E}_{\mu_{ij}(\varrho)}=\mathbb{E}_{\mu_{ij}(\varrho-1)},\quad L_{\mu_{ij}(\varrho),p}=L_{\mu_{ij}(\varrho-1),p}. \tag{3.6}$$

On the other hand, since ϱ and ϱ_{ij} are not comparable with respect to $\prec$, the blowup locus $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ is disjoint from $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$; see Subsection 2.1 above and Lemma **??** in [VZ]. Thus,

$$\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}},\quad \mathbb{E}_{\varrho}\big|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}}=\mathbb{E}_{\varrho-1}\big|_{\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}},\quad \mathcal{N}_{\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\mathcal{N}_{\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}. \tag{3.7}$$

By (3.6), (3.7), and the inductive assumptions, the three claims hold for ϱ.

Suppose that $\varrho \prec \varrho_{ij}$. Since all varieties $\overline{\mathcal{M}}_{1,\varrho'}$ intersect properly in $\overline{\mathcal{M}}_{1,(I,J)}$ in the sense of Subsection **??** in [VZ], so do their proper transforms $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho'}$ in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$. Furthermore,

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho} \subset \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}} \subset \overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$$

is the proper transform of

$$\overline{\mathcal{M}}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}_{1,\varrho} \subset \overline{\mathcal{M}}_{1,\varrho_{ij}} \subset \overline{\mathcal{M}}_{1,(I,J)}.$$

Since $\varrho\prec\varrho_{ij}$, $\mu_{ij}(\varrho-1)=\mu_{ij}(\varrho)-1$. Thus, by (3.4) and the inductive assumptions,

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho} = \overline{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,\mu_{ij}(\varrho)}\times\overline{\mathcal{M}}_{0,\{q,i,j\}} \subset \overline{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}.$$

Since $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ and $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ intersect properly, the proper transform of $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ in $\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}$, i.e. the blowup of $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ along $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$, is the blowup of $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ along $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$; see Subsection **??** in [VZ]. Thus, $\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}$ is the blowup of

$$\overline{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}$$

along $\overline{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,\mu_{ij}(\varrho)}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}$. By the construction of Subsection 2.1, this blowup is

$$\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}.$$

Furthermore, by (2.4) and the inductive assumptions,

$$\begin{aligned}\mathbb{E}_\varrho\big|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}} &= \big(\tilde{\pi}^*_\varrho\mathbb{E}_{\varrho-1}+\overline{\mathcal{M}}^{\varrho}_{1,\varrho}\big)\big|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}} = \tilde{\pi}^*_\varrho\pi^*_P\mathbb{E}_{\mu_{ij}(\varrho)-1}+\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)}\times\overline{\mathcal{M}}_{0,\{q,i,j\}}\\ &= \pi^*_P\big(\tilde{\pi}^*_{\mu_{ij}(\varrho)}\mathbb{E}_{\mu_{ij}(\varrho)-1}+\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)}\big) = \mathbb{E}_{\mu_{ij}(\varrho)}.\end{aligned}$$

We have thus verified two of the three inductive assumptions.

It remains to determine the normal bundle $\mathcal{N}_{\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}$ of $\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}$ in $\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}$. We note that by (2.4) and (3.3),

$$L_{\mu_{ij}(\varrho),p} = \begin{cases}\tilde{\pi}^*_{\mu_{ij}(\varrho)-1}L_{\mu_{ij}(\varrho)-1,p}\otimes\mathcal{O}(-\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)}), & \text{if } I_k=\{i,j\};\\ \tilde{\pi}^*_{\mu_{ij}(\varrho)-1}L_{\mu_{ij}(\varrho)-1,p}, & \text{if } I_k\supsetneq\{i,j\},\end{cases} \tag{3.8}$$

if ϱ is as in (3.2). Furthermore, if $I_k=\{i,j\}$, then

$$\overline{\mathcal{M}}_{1,\varrho}\subset\overline{\mathcal{M}}_{1,\varrho_{ij}} \qquad\Longrightarrow\qquad \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}\subset\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}.$$

Thus, by Subsection **??** in [VZ],

$$\begin{aligned}\mathcal{N}_{\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}} &= \tilde{\pi}^*_\varrho\,\mathcal{N}_{\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\otimes\mathcal{O}\big(-\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}^{\varrho}_{1,\varrho}\big)\\ &= \tilde{\pi}^*_\varrho\,\mathcal{N}_{\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\otimes\pi^*_P\mathcal{O}\big(-\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)}\big) \qquad \text{if} \quad I_k=\{i,j\}.\end{aligned} \tag{3.9}$$

On the other hand, if $I_k \supsetneq \{i,j\}$, $\overline{\mathcal{M}}_{1,\varrho}^{\varrho-1}$ and $\overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho-1}$ intersect transversally in $\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$, since $\overline{\mathcal{M}}_{1,\varrho}$ and $\overline{\mathcal{M}}_{1,\varrho_{ij}}$ intersect transversally in $\overline{\mathcal{M}}_{1,(I,J)}$. Thus,

$$\mathcal{N}_{\overline{\mathcal{M}}_{1,(I,J)}^{\varrho}} \overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho} = \tilde{\pi}_{\varrho}^{*} \, \mathcal{N}_{\overline{\mathcal{M}}_{1,(I,J)}^{\varrho-1}} \overline{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho-1} \qquad \text{if} \quad I_k \supsetneq \{i,j\}. \tag{3.10}$$

The final inductive assumption now follows from the corresponding statement for $\varrho-1$, along with (3.8)-(3.10).

Corollary 3.3 *With notation as in Lemma 2.2,*

$$\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} = \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})} \times \overline{\mathcal{M}}_{0,\{q,i,j^*\}},$$
$$\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = 0, \qquad \textit{and} \qquad \big(\tilde{f}^*\tilde{\psi}\big)|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \pi_P^*\tilde{\psi}.$$

By the paragraph preceding Proposition 2.1 and the first statement of Lemma 3.2

$$\begin{aligned}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} = \overline{\mathcal{M}}_{1,\varrho_{ij^*}}^{\varrho_{ij^*}-1} &= \overline{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}^{\mu_{ij^*}(\varrho_{ij^*}-1)} \times \overline{\mathcal{M}}_{0,\{q,i,j^*\}} \\ &= \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})} \times \overline{\mathcal{M}}_{0,\{q,i,j^*\}},\end{aligned}$$

since $\mu_{ij^*}(\varrho_{ij^*}-1)$ is the largest element of

$$\big(\mathcal{A}_1((I-\{i\})\sqcup\{p\},J-\{j^*\}),<\big),$$

according to Lemma 3.1.

Since ϱ_{ij^*} is a maximal element of $(\mathcal{A}_1(I,J),\prec)$,

$$\overline{\mathcal{M}}_{1,\varrho_{ij^*}}^{\varrho-1} \cap \overline{\mathcal{M}}_{1,\varrho}^{\varrho-1} = \emptyset \qquad \forall \varrho \in \mathcal{A}_1(I,J),\ \varrho > \varrho_{ij^*}.$$

Thus, by (2.4) and the second statement of Lemma 3.2,

$$\tilde{\mathbb{E}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \mathbb{E}_{\varrho_{ij^*}-1}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} + \sum_{\varrho \geq \varrho_{ij^*}} \overline{\mathcal{M}}_{1,\varrho}^{\varrho}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \pi_P^*\tilde{\psi} + \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}. \tag{3.11}$$

By the third statement of Lemma 3.2,

$$\begin{aligned}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \mathcal{N}_{\widetilde{\mathcal{M}}_{1,(I,J)}} \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} &= \mathcal{N}_{\overline{\mathcal{M}}_{1,(I,J)}^{\varrho_{ij^*}-1}} \overline{\mathcal{M}}_{1,\varrho_{ij^*}}^{\varrho_{ij^*}-1} \\ &= \pi_P^* L_{\mu_{ij^*}(\varrho_{ij^*}-1),p} = -\pi_P^*\tilde{\psi}.\end{aligned} \tag{3.12}$$

The second statement of Corollary 3.3 follows from (3.11) and (3.12).

Finally, by the last statement of Proposition 2.1, the second statement of Corollary 3.3, (2.12), and (3.12),

$$\big(\tilde{f}^*\tilde{\psi}\big)|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} - \sum_{i' \in I} \widetilde{\mathcal{M}}_{1,\varrho_{i'j^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = 0 - \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \pi_P^*\tilde{\psi}.$$

This concludes the proof of Corollary 3.3.

3.2 Proof of Lemma 2.3

The proof of Lemma 2.3 is analogous to the previous subsection. If I is a finite set and j, j^* are distinct elements of I, let

$$\begin{aligned}\mathcal{A}_1(I;jj^*) &= \big\{\varrho\in\mathcal{A}_1(I)-\{\varrho_{jj^*}\}: \overline{\mathcal{M}}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}_{1,\varrho}\neq\emptyset\big\}\\ &= \big\{\big(I_P,\{I_k: k\in K\}\big)\in\mathcal{A}_1(I)-\{\varrho_{jj^*}\}: \{j,j^*\}\subset I_k \text{ for some } k\in\{P\}\sqcup K\big\}.\end{aligned}$$

For each $\varrho\in\mathcal{A}_1(I;jj^*)$, we define $\eta_{jj^*}(\varrho)\in\mathcal{A}_1\big((I-\{j,j^*\})\sqcup\{p\}\big)$ by

$$\eta_{jj^*}(\varrho) = \begin{cases}\big((I_P-\{j,j^*\})\sqcup\{p\},\{I_{k'}: k'\in K\}\big), & \text{if } I_P=\{j,j^*\};\\ \big(I_P\sqcup\{p\},\{I_{k'}: k'\in K-\{k\}\}\big), & \text{if } I_k=\{j,j^*\};\\ \big(I_P,\big\{(I_k-\{j,j^*\})\sqcup\{p\}\big\}\sqcup\{I_{k'}: k'\in K-\{k\}\}\big), & \text{if } I_k\supsetneq\{j,j^*\}.\end{cases} \tag{3.13}$$

It is straightforward to see that

$$\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}_{1,\varrho} = \overline{\mathcal{M}}_{1,\eta_{jj^*}(\varrho)}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}}\subset\overline{\mathcal{M}}_{1,(I-\{j,j^*\})\sqcup\{p\}}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}}. \tag{3.14}$$

Lemma 3.4 *If I and J are finite sets, $j,j^*\in J$, and $j\neq j^*$, then the map*

$$\eta_{jj^*}: \mathcal{A}_1(I,J)\cap\mathcal{A}_1(I\sqcup J;jj^*)\longrightarrow\mathcal{A}_1\big((I,(J-\{j,j^*\})\sqcup\{p\}\big) \tag{3.15}$$

is an isomorphism of partially ordered sets.

This lemma follows easily from (2.2) and (3.13). Note, however, that it is essential that $j,j^*\in J$ and thus the second case of (3.13) does not occur if

$$\varrho\in\mathcal{A}_1(I,J)\cap\mathcal{A}_1(I\sqcup J;jj^*).$$

Lemma 3.4 implies that given an order $<$ on

$$\mathcal{A}_1\big((I,(J-\{j,j^*\})\sqcup\{p\}\big)$$

extending the partial ordering $\prec$, we can choose an order $<$ on $A_1(I,J)$ that extends the partial ordering $\prec$ such that

$$\varrho_1,\varrho_2\in\mathcal{A}_1(I,J)\cap\mathcal{A}_1(I\sqcup J;jj^*),\quad \eta_{jj^*}(\varrho_1)<\eta_{jj^*}(\varrho_2)\qquad\Longrightarrow\qquad\varrho_1<\varrho_2.$$

Below we refer to the constructions of Subsection 2.1 for the sets

$$\mathcal{A}_1\big((I,(J-\{j,j^*\})\sqcup\{p\}\big)\qquad\text{and}\qquad\mathcal{A}_1(I,J)$$

corresponding to such compatible orders $<$. We extend the map η_{jj^*} of (3.15) to $\{0\}\sqcup\mathcal{A}_1(I,J)$ by setting

$$\eta_{jj^*}(\varrho) = \begin{cases}\eta_{jj^*}(\max\{\varrho'<\varrho: \varrho'\in\mathcal{A}_1(I\sqcup J;jj^*)\}), & \text{if } \exists\,\varrho'<\varrho \text{ s.t. } \varrho'\in\mathcal{A}_1(I\sqcup J;jj^*);\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.5 *Suppose I and J are finite sets, $j,j^*\in J$, and $j\neq j^*$. If $\varrho\in\mathcal{A}_1(I,J)$, then with notation as in Subsection 2.1 and in (2.5)*

$$\pi_\varrho^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}})=\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}=\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}},\qquad \mathbb{E}_\varrho|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}}=\pi_P^*\mathbb{E}_{\eta_{jj^*}(\varrho)},$$

where

$$\pi_P:\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}}\longrightarrow\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})}$$

is the projection map onto the first component.

By (2.5) and (2.7), Lemma 3.5 holds for $\varrho=0$. Suppose $\varrho\in\mathcal{A}_1(I,J)$ and the three claims hold for $\varrho-1$. If $\varrho\notin\mathcal{A}_1(I\sqcup J,jj^*)$, then

$$\begin{gathered}\eta_{jj^*}(\varrho)=\eta_{jj^*}(\varrho-1)\qquad\Longrightarrow\\ \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})}=\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho-1)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})},\qquad \mathbb{E}_{\eta_{jj^*}(\varrho)}=\mathbb{E}_{\eta_{jj^*}(\varrho-1)}.\end{gathered}\tag{3.16}$$

On the other hand, since

$$\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}_{1,\varrho}=\emptyset,$$

the blowup locus $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ is disjoint from $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$. Thus,

$$\pi_\varrho^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}})=\pi_{\varrho-1}^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}),\quad \overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}=\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}},\quad \mathbb{E}_\varrho|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}}=\mathbb{E}_{\varrho-1}|_{\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}}.\tag{3.17}$$

By (3.16), (3.17), and the inductive assumptions, the three claims hold for ϱ.

Suppose that $\varrho\in\mathcal{A}_1(I\sqcup J,jj^*)$. Since all varieties $\overline{\mathcal{M}}_{1,\varrho'}$ intersect properly in $\overline{\mathcal{M}}_{1,(I,J)}$, so do their proper transforms $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho'}$, with $\varrho'>\varrho-1$, in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$. Since $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ is not contained in the divisor $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$, $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ and $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ intersect transversally. Thus, using the first statement of the lemma with ϱ replaced by $\varrho-1$, we obtain

$$\pi_\varrho^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}})=\tilde{\pi}_\varrho^{-1}\pi_{\varrho-1}^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}})=\tilde{\pi}_\varrho^{-1}(\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}})=\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}.$$

Furthermore,

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}\subset\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}\subset\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$$

is the proper transform of

$$\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}_{1,\varrho}\subset\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\subset\overline{\mathcal{M}}_{1,(I,J)}.$$

Since $\varrho\in\mathcal{A}_1(I\sqcup J,jj^*)$, $\eta_{jj^*}(\varrho-1)=\eta_{jj^*}(\varrho)-1$. Thus, by (3.14) and the inductive assumptions,

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}=\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,\eta_{jj^*}(\varrho)}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}}\subset\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}}.$$

Since $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ and $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ intersect properly, the proper transform of $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ in $\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}$, i.e. the blowup of $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ along $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$, is the blowup of $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ along $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$; see Subsection **??** in [VZ]. Thus, $\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}$ is the blowup of

$$\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}\times\overline{\mathcal{M}}_{0,\{q,j,j^*\}}$$

along $\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,\eta_{jj^*}(\varrho)} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}$. By the construction of Subsection 2.1, this blowup is

$$\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}.$$

Furthermore, by (2.4) and the inductive assumptions,

$$\begin{aligned}\mathbb{E}_\varrho|_{\overline{\mathcal{M}}^\varrho_{1,\varrho_{jj^*}}} &= \big(\tilde{\pi}^*_\varrho\mathbb{E}_{\varrho-1}+\overline{\mathcal{M}}^\varrho_{1,\varrho}\big)\big|_{\overline{\mathcal{M}}^\varrho_{1,\varrho_{jj^*}}} = \tilde{\pi}^*_\varrho\pi^*_P\mathbb{E}_{\eta_{jj^*}(\varrho)-1} + \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,\eta_{jj^*}(\varrho)} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}} \\ &= \pi^*_P\big(\tilde{\pi}^*_{\eta_{jj^*}(\varrho)}\mathbb{E}_{\eta_{jj^*}(\varrho)-1} + \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,\eta_{jj^*}(\varrho)}\big) = \mathbb{E}_{\eta_{jj^*}(\varrho)}.\end{aligned}$$

We have thus verified the three inductive assumptions.

Corollary 3.6 *With notation as in Proposition 2.1,*

$$\begin{aligned}\pi^{-1}\big(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\big) = \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} &\approx \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}} \\ \textit{and} \qquad \tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} &= \big(\tilde{f}^*\tilde{\psi}\big)\big|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi^*_P\tilde{\psi}.\end{aligned}$$

By Lemma 3.4, $\eta_{jj^*}(\varrho_{\max})$ is the largest element of

$$\big(\mathcal{A}_1(I,(J-\{j,j^*\})\sqcup\{p\}),<\big).$$

Thus, by the first two statements of Lemma 3.5,

$$\begin{aligned}\pi^{-1}\big(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\big) &= \pi^{-1}_{\varrho_{\max}}\big(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}\big) = \overline{\mathcal{M}}^{\varrho_{\max}}_{1,\varrho_{jj^*}} = \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \\ &= \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho_{\max})}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}} = \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}.\end{aligned}$$

By the last statement of Lemma 3.5,

$$\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \mathbb{E}_{\varrho_{\max}}\big|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi^*_P\mathbb{E}_{\eta_{jj^*}(\varrho_{\max})} = \pi^*_P\tilde{\mathbb{E}} = \pi^*_P\tilde{\psi}.$$

Finally, by the last statement of Proposition 2.1 and (2.15),

$$\big(\tilde{f}^*\tilde{\psi}\big)\big|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} - \sum_{i\in I}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}\big|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi^*_P\tilde{\psi} - 0.$$

This concludes the proof of Corollary 3.6.

3.3 Proof of Proposition 2.1

In this subsection we prove Proposition 2.1. In fact, we show that there is a lift of the forgetful map f of Proposition 2.1 to morphisms between corresponding stages of the blowup construction of Subsection 2.1 for $\overline{\mathcal{M}}_{1,(I,J)}$ and for $\overline{\mathcal{M}}_{1,(I,J-\{j^*\})}$; see Lemma 3.7 below.

First, we define a forgetful map

$$f\colon \mathcal{A}_1(I,J) \longrightarrow \bar{\mathcal{A}}_1\big(I,J-\{j^*\}\big) \equiv \mathcal{A}_1\big(I,J-\{j^*\}\big) \sqcup \big\{(I\sqcup(J-\{j^*\}),\emptyset)\big\}.$$

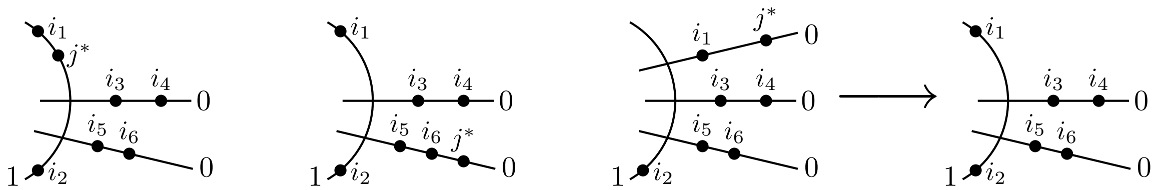


Figure 5: Images under the Forgetful Map

If $\varrho=(I_P\sqcup J_P,\{I_k\sqcup J_k\colon k\in K\})$, we put

$$f(\varrho)=\begin{cases}\big(I_P\sqcup(J_P-\{j^*\}),\{I_k\sqcup J_k\colon k\in K\}\big), & \text{if } j^*\in J_P;\\ \big(I_P\sqcup J_P,\{I_k\sqcup(J_k-\{j^*\})\}\sqcup\{I_{k'}\sqcup J_{k'}\colon k'\in K-\{k\}\}\big), & \text{if } j^*\in J_k,\ |I_k|+|J_k|>2;\\ \big((I_P\sqcup\{i\})\sqcup J_P,\{I_{k'}\sqcup J_{k'}\colon k'\in K-\{k\}\}\big), & \text{if } I_k\sqcup J_k=\{ij^*\}.\end{cases}$$

These three cases are represented in Figure 5. We note that for all $\rho\in\mathcal{A}_1(I,J-\{j^*\})$,

$$f^{-1}(\overline{\mathcal{M}}_{1,\rho})=\bigcup_{\varrho\in f^{-1}(\rho)}\overline{\mathcal{M}}_{1,\varrho}.$$

Furthermore,

$$\rho_1,\rho_2\in\bar{\mathcal{A}}_1(I,J-\{j^*\}),\quad \rho_1\neq\rho_2,\quad \varrho_1\in f^{-1}(\rho_1),\quad \varrho_2\in f^{-1}(\rho_2),\quad \varrho_1\prec\varrho_2\quad\Longrightarrow\quad\rho_1\prec\rho_2.$$

Thus, given an order $<$ on $\mathcal{A}_1(I,J-\{j^*\})$ extending the partial ordering $\prec$, we can choose an order $<$ on $\mathcal{A}_1(I,J)$ extending $\prec$ such that

$$\rho_1,\rho_2\in\bar{\mathcal{A}}_1(I,J-\{j^*\}),\quad \rho_1<\rho_2,\quad \varrho_1\in f^{-1}(\rho_1),\quad \varrho_2\in f^{-1}(\rho_2)\quad\Longrightarrow\quad\varrho_1<\varrho_2.$$

Below we will refer to the blowup constructions of Subsection 2.1 for $\overline{\mathcal{M}}_{1,(I,J)}$ and for $\overline{\mathcal{M}}_{1,(I,J-\{j^*\})}$ corresponding to such compatible orders. For each $\rho\in\mathcal{A}_1(I,J-\{j^*\})$, let

$$\rho^+=\max f^{-1}(\rho)\in\mathcal{A}_1(I,J)\qquad\text{and}\qquad\rho^-=\min f^{-1}(\rho)-1\in\{0\}\sqcup\mathcal{A}_1(I,J).$$

If ρ is not the minimal element of $\mathcal{A}_1(I,J-\{j^*\})$, then $\rho^-=(\rho-1)^+$.

Lemma 3.7 *Suppose I, J, and f are as in Proposition 2.1. For each $\rho\in\mathcal{A}_1(I,J-\{j^*\})$, f lifts to a morphism*

$$f_\rho\colon\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}\longrightarrow\overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho}$$

over the projection maps

$$\pi_{\rho^+}\colon\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}\longrightarrow\overline{\mathcal{M}}_{1,(I,J)}\qquad\textit{and}\qquad\pi_\rho\colon\overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho}\longrightarrow\overline{\mathcal{M}}_{1,(I,J-\{j^*\})};$$

see Figure 6. Furthermore,

$$f_\rho^{-1}(\overline{\mathcal{M}}_{1,\rho^*}^{\rho})=\bigcup_{\varrho\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho}^{\rho^+}\quad\forall\,\rho^*>\rho\qquad\textit{and}\qquad\mathbb{E}_{\rho^+}=f_\rho^*\mathbb{E}_\rho.\tag{3.18}$$

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+} & \xrightarrow{f_\rho} & \overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho} \\
\downarrow{\scriptstyle \pi_{\rho^+}} & & \downarrow{\scriptstyle \pi_\rho} \\
\overline{\mathcal{M}}_{1,I\sqcup J} & \xrightarrow{f} & \overline{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+} & \xrightarrow{f_\rho} & \overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho} \\
\downarrow{\scriptstyle \tilde{\pi}_{\rho^-+1}\circ\ldots\circ\tilde{\pi}_{\rho^+}} & & \downarrow{\scriptstyle \tilde{\pi}_\rho} \\
\overline{\mathcal{M}}_{1,(I,J)}^{\rho^-} & \xrightarrow{f_{\rho-1}} & \overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho-1}
\end{array}
$$

Figure 6: Main Statement of Lemma 3.7 and Inductive Step in the Proof

Proposition 2.1 follows easily from Lemma 3.7 by taking $\rho=\rho_{\max}$, where $\rho_{\max}$ is the maximal element of $\mathcal{A}_1(I,J-\{j^*\})$. We note that

$$\big\{\varrho\in\mathcal{A}_1(I,J)\colon \varrho>\rho_{\max}^+\big\}=\big\{\varrho\in\mathcal{A}_1(I,J)\colon f(\varrho)=(I\sqcup(J-\{j^*\}),\emptyset)\big\}=\big\{\varrho_{ij^*}\colon i\in I\big\}.$$

Since $\overline{\mathcal{M}}_{1,\varrho_{ij^*}}\subset\overline{\mathcal{M}}_{1,I\sqcup J}$ is a divisor for every $i\in I$, so is

$$\overline{\mathcal{M}}_{1,\varrho_{ij^*}}^{\rho_{\max}^+}\subset\overline{\mathcal{M}}_{1,I\sqcup J}^{\rho_{\max}^+}.$$

Thus, by the construction of Subsection 2.1,

$$\widetilde{\mathcal{M}}_{1,I\sqcup J}\equiv\overline{\mathcal{M}}_{1,I\sqcup J}^{\varrho_{\max}}=\overline{\mathcal{M}}_{1,I\sqcup J}^{\rho_{\max}^+}\qquad\text{and}$$

$$\mathbb{E}\equiv\mathbb{E}_{\varrho_{\max}}=\mathbb{E}_{\rho_{\max}^+}+\sum_{i\in I}\overline{\mathcal{M}}_{1,\varrho_{ij^*}}^{\rho_{\max}^+}=f_{\rho_{\max}}^*\mathbb{E}_{\rho_{\max}}+\sum_{i\in I}\overline{\mathcal{M}}_{1,\varrho_{ij^*}}^{\rho_{\max}^+}=\tilde{f}^*\mathbb{E}+\sum_{i\in I}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}},$$

where $\tilde{f}=f_{\rho_{\max}}$.

Lemma 3.7 holds for $\rho=0\in\{0\}\cup\mathcal{A}_1(I,J-\{j^*\})$, if we define $0^+=0$. Suppose

$$\rho=\big(I_P\sqcup J_P,\{I_k\sqcup J_k\colon k\in K\}\big)\in\mathcal{A}_1(I,J-\{j^*\})$$

and the lemma holds for

$$\rho-1\in\{0\}\sqcup\mathcal{A}_1(I,J-\{j^*\}).$$

The elements of $f^{-1}(\rho)\subset\mathcal{A}_1(I,J)$ can be described as follows. The largest element is

$$\rho^+=\big(I_P\sqcup(J_P\sqcup\{j^*\}),\{I_k\sqcup J_k\colon k\in K\}\big).$$

Furthermore, for each $k\in K$ and $i\in I_P$,

$$\begin{aligned}
\rho_k(j^*)&\equiv\big(I_P\sqcup J_P,\big\{I_k\sqcup(J_k\sqcup\{j^*\})\big\}\sqcup\{I_{k'}\sqcup J_{k'}\colon k'\in K-\{k\}\}\big)\in f^{-1}(\rho);\\
\rho_i(j^*)&\equiv\big((I_P-\{i\})\sqcup J_P,\big\{\{i,j\}\big\}\sqcup\{I_{k'}\sqcup J_{k'}\colon k'\in K\}\big)\in f^{-1}(\rho).
\end{aligned}$$

It is straightforward to see that

$$f^{-1}(\rho)=\big\{\rho_k(j^*)\colon k\in K\big\}\sqcup\big\{\rho_i(j^*)\colon i\in I_P\big\}\sqcup\{\rho^+\}.$$

Furthermore, ρ^+ is the largest element of $(f^{-1}(\rho),\prec)$, while no two elements of the form $\rho_k(j^*)$ and/or $\rho_i(j^*)$ are comparable with respect to $\prec$. Thus,

$$\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}\cap\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-}=\emptyset\qquad\forall\, i,k\in I_P\sqcup K,\ i\neq k;$$

see Subsection 2.1. In fact,

$$\overline{\mathcal{M}}_{1,\rho_k(j^*)}\cap\overline{\mathcal{M}}_{1,\rho_i(j^*)}=\emptyset \qquad \forall\, i,k\in I_P\sqcup K,\ i\!\neq\! k;$$

see the proof of Lemma **??** in [VZ].

All varieties $\overline{\mathcal{M}}_{1,\varrho^*}$ are smooth and intersect properly in $\overline{\mathcal{M}}_{1,(I,J)}$ in the sense of Subsection **??** in [VZ]. Thus, all varieties $\overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*}$, with $\varrho^*\!>\!\rho^-$, are also smooth and intersect properly in $\overline{\mathcal{M}}^{\rho^-}_{1,(I,J)}$. It follows that for every $k\!\in\!K$ and every point

$$p\in\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}-\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+},$$

we can choose neighborhoods $\tilde{U}$ of p in $\overline{\mathcal{M}}^{\rho^-}_{1,(I,J)}$, U of $f_{\rho-1}(p)$ in $\overline{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, and coordinates (z,v,t) on $\tilde{U}$ such that

(i) $U\!=\!f_{\rho-1}(\tilde{U})$;

(ii) $U\!=\!\big\{(z,v)\!\in\!\mathbb{C}^{|I|+|J|-|K|-1}\!\times\!\mathbb{C}^K\big\}$;

(iii) $\overline{\mathcal{M}}^{\rho-1}_{1,\rho}\!\cap\! U\!=\!\big\{(z,v)\!\in\! U:\, v\!=\!0\big\}$;

(iv) $\tilde{U}\!=\!\big\{(z,v,t)\!\in\!\mathbb{C}^{|I|+|J|-|K|-1}\!\times\!\mathbb{C}^K\!\times\!\mathbb{C}\big\}$ and $f_{\rho-1}(z,v,t)=(z,v)$.

These assumptions imply that

$$\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}\!\cap\!\tilde{U}=\big\{(z,v,t)\!\in\!\tilde{U}:\, v\!=\!0\big\}.$$

Since $\overline{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ is the blowup of $\overline{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$ along $\overline{\mathcal{M}}^{\rho-1}_{1,\rho}$, the preimage of U in $\overline{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ under the projection map is

$$V=\big\{(z,v;\ell)\!\in\! U\!\times\!\mathbb{P}(\mathbb{C}^K):\, v\!\in\!\ell\big\}.$$

Since $\overline{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ is the blowup of $\overline{\mathcal{M}}^{\rho^-}_{1,(I,J)}$ along $\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and subvarieties that do not contain p, the preimage of $\tilde{U}$ in $\overline{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ under the projection map is

$$\tilde{V}=\big\{(z,v,t;\ell)\!\in\!\tilde{U}\!\times\!\mathbb{P}(\mathbb{C}^K):\, v\!\in\!\ell\big\},$$

provided $\tilde{U}$ is sufficiently small. Thus, the map $f_{\rho-1}\colon\tilde{U}\longrightarrow U$ lifts to a map $f_\rho\colon\tilde{V}\longrightarrow V$. This lift is defined by

$$f_\rho(z,v,t;\ell)=(z,v;\ell). \tag{3.19}$$

Similarly to the previous paragraph, for every

$$p\in\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}-\bigcup_{k\in K}\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)},$$

we can choose neighborhoods $\tilde{U}$ of p in $\overline{\mathcal{M}}^{\rho^-}_{1,(I,J)}$, U of $f_{\rho-1}(p)$ in $\overline{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, and coordinates (z,v,t) on $\tilde{U}$ such that the conditions (i)-(iv) are satisfied, with $\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ replaced by $\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}$. Thus, if

$$p\notin\bigcup_{i\in I_P}\overline{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)},$$

the map $f_{\rho-1}$ lifts to the preimage of a neighborhood of p in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}$, just as in the previous paragraph.

On the other hand, suppose

$$p \in \overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-}$$

for some $i \in I_P$. Since $\overline{\mathcal{M}}_{1,\rho_i(j^*)} \subset \overline{\mathcal{M}}_{1,\rho^+}$ is of codimension-one,

$$\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-} \subset \overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$$

is also of codimension-one. We can thus choose local coordinate so that

$$\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-} \cap \tilde{U} = \big\{(z,v,t) \in \tilde{U} : v=0,\ t=0\big\}.$$

Since $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ is the blowup of $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^-}$ along $\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-}$ and subvarieties that do not contain p, the preimage of $\tilde{U}$ in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ under the projection map is

$$\tilde{V} = \big\{(z,v,t;\ell') \in \tilde{U} \times \mathbb{P}\big(\mathbb{C}^K \times \mathbb{C}\big) : (v,t) \in \ell'\big\},$$

provided $\tilde{U}$ is sufficiently small. It is immediate that

$$\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1} \cap \tilde{V} = \big\{(z,0,t;[\alpha,\beta]) \in \tilde{U} \times \mathbb{P}\big(\mathbb{C}^K \times \mathbb{C}\big) : \alpha=0\big\},$$

where $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1} \subset \overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ is the proper transform of $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$. A neighborhood of $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1} \cap \tilde{V}$ is given by

$$\tilde{U}' = \big\{(z,u,t) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^K \times \mathbb{C}\big\}, \qquad (z,u,t) \longleftrightarrow \big(z,ut,t;[u,1]\big) \in \tilde{V}.$$

Since $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}$ is the blowup of $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ along $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1}$, the preimage of $\tilde{U}$ in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}$ under the projection map is

$$\widetilde{W} = \Big(\big\{(z,u,t;\ell) \in \tilde{V}' \times \mathbb{P}\big(\mathbb{C}^K\big) : u \in \ell\big\} \cup \big\{(z,v,t;[\alpha,\beta]) \in \tilde{V} : \alpha \neq 0\big\}\Big)/\sim,$$
$$(z,u,t;\ell) \sim \big(z,ut,t;[u,1]\big).$$

Thus, the map $f_{\rho-1} : \tilde{U} \longrightarrow U$ lifts to a map $f_\rho : \widetilde{W} \longrightarrow V$. This lift is defined by

$$f_\rho(z,u,t;\ell) = (z,ut;\ell) \qquad \text{and} \qquad f_\rho\big(z,v,t;[\alpha,\beta]\big) = \big(z,v;[\alpha]\big) \tag{3.20}$$

on the two charts on $\widetilde{W}$. Note that if $u \neq 0$, then $[u] = \ell \in \mathbb{P}(\mathbb{C}^K)$. Thus, the map f_ρ agrees on the overlap of the two charts.

Finally, suppose that

$$p \in \overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-} \cap \overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$$

for some $k \in K$. Since the varieties $\overline{\mathcal{M}}_{1,\varrho^*}$ intersect properly in $\overline{\mathcal{M}}_{1,(I,J)}$, $\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}$ and $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$ intersect properly in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^-}$ and $\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-} \cap \overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$ is the proper transform of $\overline{\mathcal{M}}_{1,\rho_k(j^*)} \cap \overline{\mathcal{M}}_{1,\rho^+}$.

Thus, $\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}\cap\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$ is a divisor in $\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}$ and in $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$. If follows that we can choose neighborhoods $\tilde{U}$ of p in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^-}$, U of $f_{\rho-1}(p)$ in $\overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho-1}$, and coordinates (z,v,w_k,w_+) on $\tilde{U}$ such that

(i) $U=f_{\rho-1}(\tilde{U})$;

(ii) $U=\big\{(z,v,w)\in\mathbb{C}^{|I|+|J|-|K|-1}\times\mathbb{C}^{K-\{k\}}\times\mathbb{C}\big\}$;

(iii) $\overline{\mathcal{M}}_{1,\rho}^{\rho-1}\cap U=\big\{(z,v,w)\in U:\, v=0,\; w=0\big\}$;

(iv) $\tilde{U}=\big\{(z,v,w_k,w_+)\in\mathbb{C}^{|I|+|J|-|K|-1}\times\mathbb{C}^{K-\{k\}}\times\mathbb{C}\times\mathbb{C}\big\}$, $f_{\rho-1}(z,v,w_k,w_+)=(z,v,w_kw_+)$;

(v) $\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}\cap\tilde{U}=\big\{(z,v,w_k,w_+)\in\tilde{U}:\; v=0,\; w_+=0\big\}$;

(vi) $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}\cap\tilde{U}=\big\{(z,v,w_k,w_+)\in\tilde{U}:v=0,\; w_k=0\big\}$.

Similarly to the above, the preimage of U in $\overline{\mathcal{M}}_{1,(I,J-\{j^*\})}^{\rho}$ under the projection map is

$$V=\big\{(z,v,w;\ell)\in U\times\mathbb{P}\big(\mathbb{C}^{K-\{k\}}\times\mathbb{C}\big):(v,w)\in\ell\big\}.$$

Since $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ is the blowup of $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^-}$ along $\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}$ and subvarieties that do not contain p, the preimage of $\tilde{U}$ in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ under the projection map is

$$\tilde{V}=\big\{(z,v,w_k,w_+;\ell_k)\in\tilde{U}\times\mathbb{P}\big(\mathbb{C}^{K-\{k\}}\times\mathbb{C}\big):(v,w_+)\in\ell_k\big\},$$

provided $\tilde{U}$ is sufficiently small. It is immediate that

$$\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1}\cap\tilde{V}=\big\{(z,0,0,w_+;[\alpha,\beta])\in\tilde{U}\times\mathbb{P}\big(\mathbb{C}^{K-\{k\}}\times\mathbb{C}\big):\alpha=0\big\},$$

where $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1}\subset\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ is the proper transform of $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}$. A neighborhood of $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1}\cap\tilde{V}$ is given by

$$\begin{gathered}\tilde{U}'=\big\{(z,u,u_k,w_+)\in\mathbb{C}^{|I|+|J|-|K|-1}\times\mathbb{C}^{K-\{k\}}\times\mathbb{C}\times\mathbb{C}\big\},\\ (z,u,u_k,w_+)\longleftrightarrow\big(z,uw_+,u_k,w_+;[u,1]\big)\in\tilde{V}.\end{gathered}$$

Since $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}$ is the blowup of $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+-1}$ along $\overline{\mathcal{M}}_{1,\rho^+}^{\rho^+-1}$, the preimage of $\tilde{U}$ in $\overline{\mathcal{M}}_{1,(I,J)}^{\rho^+}$ under the projection map is

$$\begin{aligned}\widetilde{W}=\big(&\big\{(z,u,u_k,w_+;\ell)\in\tilde{V}'\times\mathbb{P}\big(\mathbb{C}^{K-\{k\}}\times\mathbb{C}\big):(u,u_k)\in\ell\big\}\\ &\qquad\cup\big\{(z,v,w_k,w_+;[\alpha,\beta])\in\tilde{V}:\alpha\neq0\big\}\big)/\sim,\\ &(z,u,u_k,w_+;\ell)\sim\big(z,uw_+,u_k,w_+;[u,1]\big).\end{aligned}$$

Thus, the map $f_{\rho-1}\colon\tilde{U}\longrightarrow U$ lifts to a map $f_\rho\colon\widetilde{W}\longrightarrow V$. This lift is defined by

$$\begin{aligned}f_\rho(z,u,u_k,w_+;\ell)&=(z,uw_+,u_kw_+;\ell)\qquad\text{and}\\ f_\rho\big(z,v,w_k,w_+;[\alpha,\beta]\big)&=(z,v,w_kw_+;[\alpha,w_k\beta])\end{aligned}\tag{3.21}$$

on the two charts on $\widetilde{W}$. It is immediate that f_ρ is well-defined on the overlap of the two charts.

Remark: The first equality in (3.18) should be viewed as incorporating the above information concerning the local structure of the projection map. It is easy to see from the verification of the first

equality in (3.18) below that this additional information is preserved by the inductive step as well.

It remains to verify that the two equalities in (3.18) still hold. Let

$$\pi_{\rho,\rho-1}\colon \overline{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \longrightarrow \overline{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})} \qquad \text{and}$$
$$\pi_{\rho^+,\varrho}\colon \overline{\mathcal{M}}^{\rho^+}_{1,(I,J)} \longrightarrow \overline{\mathcal{M}}^{\varrho}_{1,(I,J)}, \quad \varrho\in\{\rho^-\}\cup f^{-1}(\rho)$$

be the projection maps. By the construction of the line bundles $\mathbb{E}_\varrho$ in Subsection 2.1,

$$\mathbb{E}_\rho = \pi^*_{\rho,\rho-1}\mathbb{E}_\rho + \overline{\mathcal{M}}^{\rho}_{1,\rho} \qquad \text{and} \tag{3.22}$$
$$\mathbb{E}_{\rho^+} = \pi^*_{\rho^+,\rho^-}\mathbb{E}_{\rho^-} + \sum_{\varrho\in f^{-1}(\rho)} \pi^*_{\rho^+,\varrho}\overline{\mathcal{M}}^{\varrho}_{1,\varrho} = \pi^*_{\rho^+,\rho^-}\mathbb{E}_{\rho^-} + \sum_{\varrho\in f^{-1}(\rho)} \pi^{-1}_{\rho^+,\varrho}\big(\overline{\mathcal{M}}^{\varrho}_{1,\varrho}\big), \tag{3.23}$$

where

$$\overline{\mathcal{M}}^{\rho}_{1,\rho} = \pi^{-1}_{\rho,\rho-1}\big(\overline{\mathcal{M}}^{\rho-1}_{1,\rho}\big) \subset \overline{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \qquad \text{and} \qquad \overline{\mathcal{M}}^{\varrho}_{1,\varrho} \subset \pi^{-1}_{\varrho,\varrho-1}\big(\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}\big)$$

are the exceptional divisors for the blowups at the steps ρ and ϱ. Since all divisors $\pi^{-1}_{\rho^+,\varrho}\big(\overline{\mathcal{M}}^{\varrho}_{1,\varrho}\big)$ are distinct,

$$\begin{aligned}\sum_{\varrho\in f^{-1}(\rho)} \pi^{-1}_{\rho^+,\varrho}\big(\overline{\mathcal{M}}^{\varrho}_{1,\varrho}\big) &= \pi^{-1}_{\rho^+,\rho^-}\bigg(\bigcup_{\varrho\in f^{-1}(\rho)}\overline{\mathcal{M}}^{\rho^-}_{1,\varrho}\bigg) = \pi^{-1}_{\rho^+,\rho^-}\big(f^{-1}_{\rho-1}\big(\overline{\mathcal{M}}^{\rho-1}_{1,\rho}\big)\big)\\ &= f^{-1}_{\rho}\pi^{-1}_{\rho,\rho-1}\big(\overline{\mathcal{M}}^{\rho-1}_{1,\rho}\big) = f^{-1}_{\rho}\big(\overline{\mathcal{M}}^{\rho}_{1,\rho}\big) = f^{*}_{\rho}\big(\overline{\mathcal{M}}^{\rho}_{1,\rho}\big).\end{aligned} \tag{3.24}$$

The second equality in (3.18) follows from the same equality with ρ replaced by $\rho-1$, along with (3.22)-(3.24).

Suppose next that $\rho^* > \rho$. Since

$$\pi_{\rho,\rho-1}\circ f_\rho = f_{\rho-1}\circ\pi_{\rho^+,\rho^-},$$

$\overline{\mathcal{M}}^{\rho}_{1,\rho^*}$ is the proper transform of $\overline{\mathcal{M}}^{\rho-1}_{1,\rho^*}$, and $\overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*}$ is the proper transform of $\overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*}$,

$$f^{-1}_{\rho}\big(\overline{\mathcal{M}}^{\rho}_{1,\rho^*}\big) \supset \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*}$$

by the first equation in (3.18) with ρ replaced by $\rho-1$. We will next verify the opposite inclusion. Suppose

$$q\in\overline{\mathcal{M}}^{\rho}_{1,\rho^*}, \qquad \tilde{p}\in f^{-1}_{\rho}(q), \qquad \text{and}$$
$$p=\pi_{\rho^+,\rho^-}(\tilde{p})\in f^{-1}_{\rho-1}\big(\overline{\mathcal{M}}^{\rho-1}_{1,\rho^*}\big) = \bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*} \subset \overline{\mathcal{M}}^{\rho^-}_{1,(I,J)}.$$

If $\pi_{\rho,\rho-1}(q)\notin\overline{\mathcal{M}}^{\rho-1}_{1,\rho}$, then

$$f^{-1}_{\rho}(q) = f^{-1}_{\rho-1}\big(\pi_{\rho,\rho-1}(q)\big) = p \in \bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*} - \bigcup_{\varrho\in f^{-1}(\rho)}\overline{\mathcal{M}}^{\rho^-}_{1,\varrho} \subset \bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*},$$

as needed.

Suppose that

$$\pi_{\rho,\rho-1}(q) \in \overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)} \equiv \overline{\mathcal{M}}^{\rho-1}_{1,\rho} \cap \overline{\mathcal{M}}^{\rho-1}_{1,\rho^*}.$$

First, we consider the case when

$$p \in \overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} - \overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}$$

for some $k \in K$. Since $\overline{\mathcal{M}}^{\rho-1}_{1,\rho}$ and $\overline{\mathcal{M}}^{\rho-1}_{1,\rho^*}$ intersect properly in $\overline{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, we can choose local coordinates (z,v,t) near p as in the first case considered above such that for some $K_{\rho^*} \subset K$

(v) $\overline{\mathcal{M}}^{\rho-1}_{1,\rho^*} \cap U = \big\{(z,v) \in U : z \in \overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v \in \mathbb{C}^{K_{\rho^*}}\big\}$.

This assumption implies that

$$\overline{\mathcal{M}}^{\rho}_{1,\rho} \cap \overline{\mathcal{M}}^{\rho}_{1,\rho^*} \cap V = \big\{(z,0;\ell) \in V : z \in \overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}})\big\}. \tag{3.25}$$

In addition, by (iv) and the structure of $f_{\rho-1}$,

$$\bigcup_{\varrho^* \in f^{-1}(\rho^*)} \overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*} \cap \tilde{U} = f^{-1}_{\rho-1}\big(\overline{\mathcal{M}}^{\rho-1}_{1,\rho^*}\big) \cap \tilde{U} = \big\{(z,v,t) \in \tilde{U} : z \in \overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v \in \mathbb{C}^{K_{\rho^*}}\big\}.$$

Since $\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and $\overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*}$ intersect properly, it follows that

$$\bigcup_{\varrho^* \in f^{-1}(\rho^*)} \overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*} \cap \overline{\mathcal{M}}^{\rho^+}_{1,\rho_k(j^*)} \cap \tilde{V} = \big\{(z,0,t;\ell) \in \tilde{V}; z \in \overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}})\big\}.$$

Using (3.19), we conclude that

$$\tilde{p} \in \big\{f_\rho|_{\tilde{V}}\big\}^{-1}\big(\overline{\mathcal{M}}^{\rho}_{1,\rho^*} \cap \overline{\mathcal{M}}^{\rho}_{1,\rho}\big) = \bigcup_{\varrho^* \in f^{-1}(\rho^*)} \overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*} \cap \overline{\mathcal{M}}^{\rho^+}_{1,\rho_k(j^*)} \cap \tilde{V},$$

as needed.

Suppose next that

$$p \in \overline{\mathcal{M}}^{\rho^-}_{1,\rho^+} - \bigcup_{k \in K} \overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)},$$

i.e. as in the second case considered above. We can again choose $K_{\rho^*} \subset K$ so that Condition (v) in the previous paragraph is satisfied. If

$$p \notin \bigcup_{i \in I_P} \overline{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)},$$

then the same argument as in the previous paragraph, but with replaced $\rho_k(j^*)$ by ρ^+, shows that

$$\tilde{p} \in \bigcup_{\varrho^* \in f^{-1}(\rho^*)} \overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*}.$$

On the other hand, suppose that

$$p \in \overline{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}$$

for some $i\in I_P$. Then, with notation as in the construction of the map f_ρ in this case,

$$\begin{aligned}
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+-1}\cap\tilde{V}&=\big\{(z,v,t;\ell')\in\tilde{V}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell'\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\big\} \implies\\
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+-1}\cap\tilde{U}'&=\big\{(z,u,t)\in\tilde{U}': z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ u\in\mathbb{C}^{K_{\rho^*}}\big\} \implies\\
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+}\cap\pi_{\rho^+,\rho^-}^{-1}\big(\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-}\big)\cap\widetilde{W}&=\big\{(z,u,0;\ell)\in\widetilde{W}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}})\subset\mathbb{P}\big(\mathbb{C}^K\big)\big\}\\
&\quad\cup\big\{(z,0,0;\ell')\in\widetilde{W}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell'\in\mathbb{P}\big(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C}\big)\subset\mathbb{P}\big(\mathbb{C}^K\times\mathbb{C}\big)\big\}.
\end{aligned}$$

Using (3.20) and (3.25), we conclude that

$$\tilde{p}\in\big\{f_\rho|_{\pi_{\rho^+,\rho^-}^{-1}(\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-})\cap\widetilde{W}}\big\}^{-1}\big(\overline{\mathcal{M}}_{1,\rho^*}^{\rho}\cap\overline{\mathcal{M}}_{1,\rho}^{\rho}\big)=\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+}\cap\pi_{\rho^+,\rho^-}^{-1}\big(\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-}\big)\cap\widetilde{W}.$$

Note that the map $f_\rho|_{\pi_{\rho^+,\rho^-}^{-1}(\overline{\mathcal{M}}_{1,\rho_i(j^*)}^{\rho^-})\cap\widetilde{W}}$ is a $\mathbb{P}^1$-fibration, while the map $f_\rho|_{\tilde{V}}$ of the previous paragraph is a $\mathbb{C}$-fibration.

Finally, suppose that

$$p\in\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}\cap\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}$$

for some $k\in K$. With notation as in the corresponding case in the construction of the map f_ρ and with a good choice of local coordinates, we have two cases to consider. There exists $K_{\rho^*}\subset K-\{k\}$ such that

Case 1: $\overline{\mathcal{M}}_{1,\rho^*}^{\rho-1}\cap U=\big\{(z,v,w)\in U: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ v\in\mathbb{C}^{K_{\rho^*}}\big\}$;

Case 2: $\overline{\mathcal{M}}_{1,\rho^*}^{\rho-1}\cap U=\big\{(z,v,w)\in U: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ v\in\mathbb{C}^{K_{\rho^*}},\ w=0\big\}$.

In the first case, we have

$$\begin{aligned}
\overline{\mathcal{M}}_{1,\rho^*}^{\rho}\cap\overline{\mathcal{M}}_{1,\rho}^{\rho}\cap V&=\big\{(z,0,0;\ell)\in V: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\big\}\qquad\text{and}\\
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^-}\cap\tilde{U}&=f_{\rho-1}^{-1}\big(\overline{\mathcal{M}}_{1,\rho^*}^{\rho-1}\big)\cap\tilde{U}=\big\{(z,v,w_k,w_+)\in\tilde{U}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ v\in\mathbb{C}^{K_{\rho^*}}\big\}.
\end{aligned}\tag{3.26}$$

It follows that

$$\begin{aligned}
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+-1}\cap\tilde{V}&=\big\{(z,v,w_k,w_+;\ell_k)\in\tilde{V}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell_k\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\big\} \implies\\
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+-1}\cap\tilde{U}'&=\big\{(z,u,u_k,w_+)\in\tilde{U}': z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ u\in\mathbb{C}^{K_{\rho^*}}\big\} \implies\\
\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}_{1,\varrho^*}^{\rho^+}&\cap\pi_{\rho^+,\rho^-}^{-1}\big(\overline{\mathcal{M}}_{1,\rho^+}^{\rho^-}\cap\overline{\mathcal{M}}_{1,\rho_k(j^*)}^{\rho^-}\big)\cap\widetilde{W}\\
&=\big\{(z,u,0,0;\ell)\in\widetilde{W}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell\in\mathbb{P}\big(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C}\big)\big\}\\
&\qquad\cup\big\{(z,0,0,0;\ell_k)\in\widetilde{W}: z\in\overline{\mathcal{M}}_{(\rho,\rho^*)}^{\rho-1};\ \ell_k\in\mathbb{P}\big(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C}\big)\big\}.
\end{aligned}$$

Thus, by (3.21) and (3.26),

$$\begin{aligned}\tilde{p}\in\big\{f_\rho|_{\pi^{-1}_{\rho^+,\rho^-}(\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)})\cap\widetilde{W}}\big\}^{-1}\big(\overline{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\overline{\mathcal{M}}^{\rho}_{1,\rho}\big)\\ =\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*}\cap\pi^{-1}_{\rho^+,\rho^-}\big(\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}\big)\cap\widetilde{W}.\end{aligned}\tag{3.27}$$

In the second case above,

$$\begin{gathered}\overline{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\overline{\mathcal{M}}^{\rho}_{1,\rho}\cap V=\big\{(z,0,0;\ell)\in V\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times 0)\big\}\qquad\text{and}\\ \bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}^{\rho^-}_{1,\varrho^*}\cap\tilde{U}=f^{-1}_{\rho-1}\big(\overline{\mathcal{M}}^{\rho-1}_{1,\rho^*}\big)\cap\tilde{U}=\tilde{\mathcal{Z}}^{\rho^-}_k\cup\tilde{\mathcal{Z}}^{\rho^-}_+,\qquad\text{where}\\ \tilde{\mathcal{Z}}^{\rho^-}_{\circledast}=\big\{(z,v,w_k,w_+)\in\tilde{U}\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v\in\mathbb{C}^{K_{\rho^*}},\ w_{\circledast}=0\big\},\qquad\circledast=k,+.\end{gathered}\tag{3.28}$$

We denote by $\tilde{\mathcal{Z}}^{\rho^+-1}_k$ and $\tilde{\mathcal{Z}}^{\rho^+-1}_+$ the proper transforms of $\tilde{\mathcal{Z}}^{\rho^-}_k$ and $\tilde{\mathcal{Z}}^{\rho^-}_+$ in $\tilde{V}$ and by $\tilde{\mathcal{Z}}^{\rho^+}_k$ and $\tilde{\mathcal{Z}}^{\rho^+}_+$ the proper transforms of $\tilde{\mathcal{Z}}^{\rho^-}_k$ and $\tilde{\mathcal{Z}}^{\rho^-}_+$ in $\widetilde{W}$. Then,

$$\begin{aligned}\tilde{\mathcal{Z}}^{\rho^+-1}_k&=\big\{(z,v,0,w_+;\ell_k)\in\tilde{V}\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\big\}\qquad\Longrightarrow\\ \tilde{\mathcal{Z}}^{\rho^+-1}_k\cap\tilde{U}'&=\big\{(z,u,0,w_+,)\in\tilde{U}'\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ u\in\mathbb{C}^{K_{\rho^*}}\big\}\qquad\Longrightarrow\\ \tilde{\mathcal{Z}}^{\rho^+}_k\cap\pi^{-1}_{\rho^+,\rho^-}\big(\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}\big)&=\big\{(z,u,0,0;\ell)\in\widetilde{W}\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times 0)\big\}\\ &\quad\cup\big\{(z,0,0,0;\ell_k)\in\widetilde{W}\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k\in\mathbb{P}\big(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C}\big)\big\}.\end{aligned}\tag{3.29}$$

Similarly,

$$\begin{aligned}&\tilde{\mathcal{Z}}^{\rho^+-1}_+=\big\{(z,v,w_k,0;\ell_k)\in\tilde{V}\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times 0)\big\}\quad\Longrightarrow\quad\tilde{\mathcal{Z}}^{\rho^+-1}_+\cap\tilde{U}'=\emptyset\quad\Longrightarrow\\ &\tilde{\mathcal{Z}}^{\rho^+}_+\cap\pi^{-1}_{\rho^+,\rho^-}\big(\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}\big)=\big\{(z,0,0,0;\ell_k)\in\widetilde{W}\colon z\in\overline{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k\in\mathbb{P}\big(\mathbb{C}^{K_{\rho^*}}\times 0\big)\big\}.\end{aligned}\tag{3.30}$$

Since

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\overline{\mathcal{M}}^{\rho^+}_{1,\varrho^*}\cap\pi^{-1}_{\rho^+,\rho^-}\big(\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}\big)\cap\widetilde{W}=\big(\tilde{\mathcal{Z}}^{\rho^+}_k\cap\tilde{\mathcal{Z}}^{\rho^+}_+\big)\cap\pi^{-1}_{\rho^+,\rho^-}\big(\overline{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\overline{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}\big),$$

we conclude from (3.21) and (3.28)-(3.30) that (3.27) holds in this case as well.

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